BMS-like structures in cosmology

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Asymptotic symmetries in cosmological spacetimes

but first some words about asymptotically flat spacetimes...
From messy physics to peaceful real
Key idea: bring infinity to a finite distance

\((\hat{M}, \hat{g}_{ab})\) \hspace{1cm} \((M, g_{ab} = \Omega^2 \hat{g}_{ab})\)

Conformal completion
Asymptotic flatness

A physical spacetime \((\hat{M}, \hat{g}_{ab})\) is asymptotically flat if there exists a spacetime \((M, g_{ab})\) with boundary \(\partial M \cong \mathcal{I} \cong \mathbb{R} \times S^2\) such that

1. \(\Omega\) and \(g_{ab} = \Omega^2 \hat{g}_{ab}\) are smooth on \(M\), \(\Omega \equiv 0\) and \(n_a = \nabla_a \Omega\) is nowhere vanishing on \(\mathcal{I}\)

2. Einstein’s equations are satisfied with \(\hat{T}_{ab}\) such that \(\Omega^{-2} \hat{T}_{ab}\) has a smooth limit to \(\mathcal{I}\)
Consequences

- Einstein’s equation \( n^a \) is null on \( \mathcal{I} \)
- \( q_{ab} = \) induced metric on \( \mathcal{I} \) is degenerate: 0 ++

Conformal freedom

\[
\Omega \rightarrow \Omega' = \omega \Omega
\]

\[
g_{ab} \rightarrow g'_{ab} = \omega^2 g_{ab}
\]

\[
n^a \rightarrow n'^a = \omega^{-1} n^a
\]
Universal structure

This is common to \textit{all} asymptotically flat spacetimes

\[
\{ q_{ab}, n^a \} = \oint \omega^2 q_{ab}, \omega^{-1} n^a
\]

Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime
Example: flat spacetime

\[ ds^2 = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right) \]

\[ ds^2 = -du^2 + 2 du \ dr + r^2 \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right) \]

\[ ds^2 = -\Omega^2 \ du^2 + 2 \ du \ d\Omega + d\theta^2 + \sin^2 \theta \ d\phi^2 \]

\[ ds^2 \equiv 2 \ du \ d\Omega + d\theta^2 + \sin^2 \theta \ d\phi^2 \]

Others examples: Kerr-Newman spacetimes, Weyl spacetimes, etc.
Generic asymptotically flat spacetime

\[ ds^2 = -UV \, du^2 - 2U \, dudr + \gamma_{AB}(r \, d\theta^A + W^A \, du)(r \, d\theta^B + W^B \, du) \]

\[ U = 1 + B/r^2 + O(r^{-3}), \]
\[ V = 1 - 2M/r + N/r^2 + O(r^{-3}), \]
\[ W^A = A^A/r + B^A/r^2 + O(r^{-3}), \]
\[ \gamma_{AB} = \Omega_{AB} + f_{AB}/r + \frac{1}{4} f^2 \Omega_{AB}/r^2 + O(r^{-3}) \]
Generic asymptotically flat spacetime

\[ ds^2 = -UV \, du^2 - 2U \, du \, dr + \gamma_{AB} (r \, d\theta^A + W^A \, du)(r \, d\theta^B + W^B \, du) \]

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Two definitions are equivalent!

Geometric description à la Penrose (with the conformal completion)

Coordinate description à la Bondi & Sachs
Asymptotic symmetry algebra

“Spacetime diffeomorphism that leave the universal structure at scri invariant”

Universal structure
\[ \mathcal{L}_\xi q_{ab} \equiv 2 \alpha q_{ab} \text{ with } \mathcal{L}_n \alpha \equiv 0 \]
\[ \mathcal{L}_\xi n^a \equiv -\alpha n^a \]

Coordinates
\[ \Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \equiv 0 \]
**Bondi-Metzner-Sachs algebra (BMS)**

- **Bigger** than Poincare (=translations & rotations)
- BMS = supertranslations & rotations

\[
\xi^a \partial_a = \left( f(\theta, \varphi) \right) + \frac{1}{2} u D_A Y^A \partial_u + Y^A \partial_A
\]

- **supertranslations**
- **rotations**
  \[2D_{(A} Y_{B)} + q_{AB} D_C Y^C = 0\]
What is BMS good for?

It provides quantities with a physical interpretation!
Critical assumption

Move far away from sources: ‘spacetime becomes flat’
Expanding spacetimes are not asymptotically flat!
Why assume asymptotic flatness?

P. G. Bergmann:

The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. Bondi:

I regret it as much as you do, that we haven’t yet got to the point of doing the Friedmann universe.
Today: NO cosmological constant
Decelerating FLRW spacetimes

\[ ds^2 = a^2(\eta) \left[ -d\eta^2 + dr^2 + r^2 d\Omega^2 + \sum_{i=1}^{3} dx_i^2 \right] \]

Physical metric
\[ a(\eta) = \left( \frac{\eta}{\eta_0} \right)^{\frac{1}{3-s}} \]

\[ S = \frac{2}{3(1+w)} \]
\[ 0 \leq S < 1 \]
\[ -\frac{1}{3} < w < \infty \]

\[ P = w \rho \]

- \( w = 1 \) stiff fluid
- \( w = 1/3 \) radiation
- \( w = 0 \) dust
- \( w = -1 \) cosmological constant
\[ ds^2 = \alpha^2(\eta) \left[-d\eta^2 + dr^2 + r^2 S_{AB} dx^A dx^B \right] \]

\[
\begin{align*}
\eta &= \frac{\sin T}{\cos R + \cos T} \\
&= \frac{\sin \left( \frac{V+U}{2} \right)}{2 \cos \frac{U}{2} \cos \frac{V}{2}} \\
&= \frac{\sin \left( \frac{V-U}{2} \right)}{2 \cos \frac{U}{2} \cos \frac{V}{2}} \\
\end{align*}
\]

\[ r = \frac{\sin R}{\cos R + \cos T} \]

\[
\begin{align*}
\{ U = T-R \} & \iff -\pi < U < \pi \\
\{ V = T+R \} & \iff 0 < V < \pi \\
\end{align*}
\]

Choose \( \Omega = 2 \left( \cos \frac{U}{2} \cos \frac{V}{2} \right)^{1/3} \left( \sin \left( \frac{U+V}{2} \right) \right)^{-\frac{1}{3}} \)

\[ ds^2 = \Omega^2 ds^2 = -dU dV + \sin \left( \frac{V-U}{2} \right)^2 S_{AB} dx^A dx^B \]

\[ \Rightarrow \text{can add } V = -U \text{ & } V = \pi, \text{ because this metric is smooth everywhere including at the boundaries} \]
\[ i^+ = \{ V = U = \pi y \} \]

\[ \gamma = \{ V = \pi y \} \]

\[ \text{Big Bang} = \{ V = -U = \pi y \} \]

\[ i^- = \{ V = -U = \pi y \} \]
\[ i^+ = \{ V = U = \pi \} \]

\[ y = \{ V = \pi \} \]

\[ \omega = 0 \text{ on all of } y \]

Big Bang = \{ V = -U \}

\[ \omega \text{ diverges here} \]
The conformal factor

But near $y$...

$$\Omega \sim \cos \frac{\theta}{2} (\pi - \psi)^{1/s}$$

$$\nabla_a \Omega \sim \cos \frac{\theta}{2} (\pi - \psi)^{1/s} \nabla_a \psi \rightarrow \hat{\nabla}_a = 0 \text{ unless } \theta = 0$$

Bad choice for $\Omega$?

What to do?

$$\Omega' = \omega \Omega$$
with $$\omega \sim (\pi - \psi)^{-\frac{1}{1-s}} \Rightarrow \Omega' \text{ is smooth} \at \gamma \checkmark$$

$$\nabla_a \Omega' \neq 0 \checkmark$$

but then

$$g^a_{\dot{b}} = \Omega^2 \frac{\hat{g}_{\dot{a} \dot{b}}}{(\pi - \psi)^{2(1-s)}} \nabla_a \psi$$

THIS DIVERGES @ $\gamma$!
Simple resolution

\( \Omega^{1-s} \) is smooth @ \( G \)

* \( \Omega^{1-s} \equiv 0 \)

* \( \nabla_a \Omega^{1-s} \neq 0 \)

Define the normal to \( y \) using \( \Omega^{1-s} \)

\[
\Rightarrow \ n_x = \frac{1}{1-s} \nabla_a \Omega^{1-s} = \Omega^{1-s} \nabla_a \Omega = -2^{-s} (c \cos \frac{y}{2})^{1-s} \nabla_a V
\]
Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2} \hat{T}_{ab}$ should have a limit to $\mathcal{I}$ but FLRW spacetimes are homogeneous, so there is matter everywhere!

$$\lim_{z \to 0} 8\pi G \ g^{ab} \hat{T}_{ab} = \frac{68(1-s)}{(1-s)^2} \left( \sec \frac{\Omega}{2} \right)^2 \rightarrow \text{NON-VANISHING}$$

$$8\pi G \ \hat{T}_{ab} = 28 \Omega^{2(1-s)} \ n_a n_b + 28 \Omega^{2-s} \ \tau_{a} n_{b} \ + \text{finite}$$

universal

depends on choice $\Omega$

$\tau_{a} \approx \tan \frac{\Omega}{2} (\rho U + \rho V)$
Spacetimes with a cosmological null asymptote

A physical spacetime \((\widehat{M}, \hat{g}_{ab})\) admits a cosmological null asymptote if there exists a spacetime \((M, g_{ab})\) with boundary \(\partial M \cong \mathcal{I} \cong \mathbb{R} \times S^2\) such that

1. \(\Omega \cong 0, \Omega^{1-s}\) and \(g_{ab} = \Omega^2 \hat{g}_{ab}\) is smooth on \(M\),
   \[n_a = \Omega^{-s} \nabla_a \Omega\] is nowhere vanishing on \(\mathcal{I}\) (for \(0 \leq s < 1\))

2. Einstein’s equations are satisfied with \(\hat{T}_{ab}\) such that
   \[
   \lim_{\mathcal{I} \to \mathcal{I}} g^{ab} \hat{T}_{ab} \quad \text{exists}
   \]
   \[
   \lim_{\mathcal{I} \to \mathcal{I}} \Omega^{1-s} \left[ 8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)}n_an_b \right] \cong 2s\tau(a)n_b
   \]
Spacetimes with a cosmological null asymptote

A physical spacetime \((\mathcal{M}, \hat{g}_{ab})\) admits a cosmological null asymptote if there exists a spacetime \((M, g_{ab})\) with boundary \(\partial M \cong I \cong \mathbb{R} \times S^2\) such that

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   \[
   \lim_{\mathcal{I} \to \mathcal{J}} \Omega^{1-s} \left[ 8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)} n_a n_b \right] \cong 2s\tau_{(a} n_{b)}
   \]
Asymptotic symmetry algebra

All smooth vector fields that map

\[ \{ q_{ab}, n^a \} \rightarrow \{ q_{\hat{a}\hat{b}} = \omega^2 q_{ab}, n'{}^a = \omega^{-1-s} n^a \} \]

\[ \Rightarrow b_s \cong SO(1,3) \times \mathbb{R}_s \]

- Lorentz subalgebra
- \( s \)-dependent super translations
Didn’t we know this already?

BMS in Cosmology

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Abstract

Symmetries play an interesting role in cosmology. They are useful in characterizing the cosmological perturbations generated during inflation and lead to consistency relations involving the
There is a twist!

Supertranslations

$\xi^a = f(\theta, \phi) n^a$

$\triangleright$ $f$ has conformal weight $1 + s$

Who cares?

\[\text{No translation subalgebra}\]
Not exactly BMS in cosmology

BMS-like symmetries in cosmology

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Abstract

Null infinity in asymptotically flat spacetimes possess a rich mathematical structure; including the BMS group and the Bondi news tensor that allow one to study gravitational radiation rigorously. However, FLRW spacetimes are not asymptotically flat because their stress-energy tensor does not decay sufficiently fast and in fact diverges at null infinity. This class includes matter- and radiation-dominated FLRW spacetimes. We define a class of spacetimes whose structure at null infinity is similar to FLRW spacetimes: the stress-energy tensor is allowed to diverge and the notion of mass and linear momentum?
Any other examples?

Class of spacetimes at least as big as asymptotically flat spacetimes!

Linearization stability still open question
Geometric construction to study spacetimes beyond asymptotic flatness in the cosmological context

Asymptotic symmetry algebra is BMS-like

➢ It does not have a translation subalgebra!
Future applications

❖ Next order structure

➢ Study rigorously gravitational radiation produced by compact sources in cosmological spacetimes
➢ Study the gravitational memory effect
➢ Charges and fluxes

❖ Link with timelike future infinity

❖ … your favorite topic!