BMS-like structures in cosmology

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ICERM Workshop “Advances and Challenges in Computational Relativity” – September 15 2020

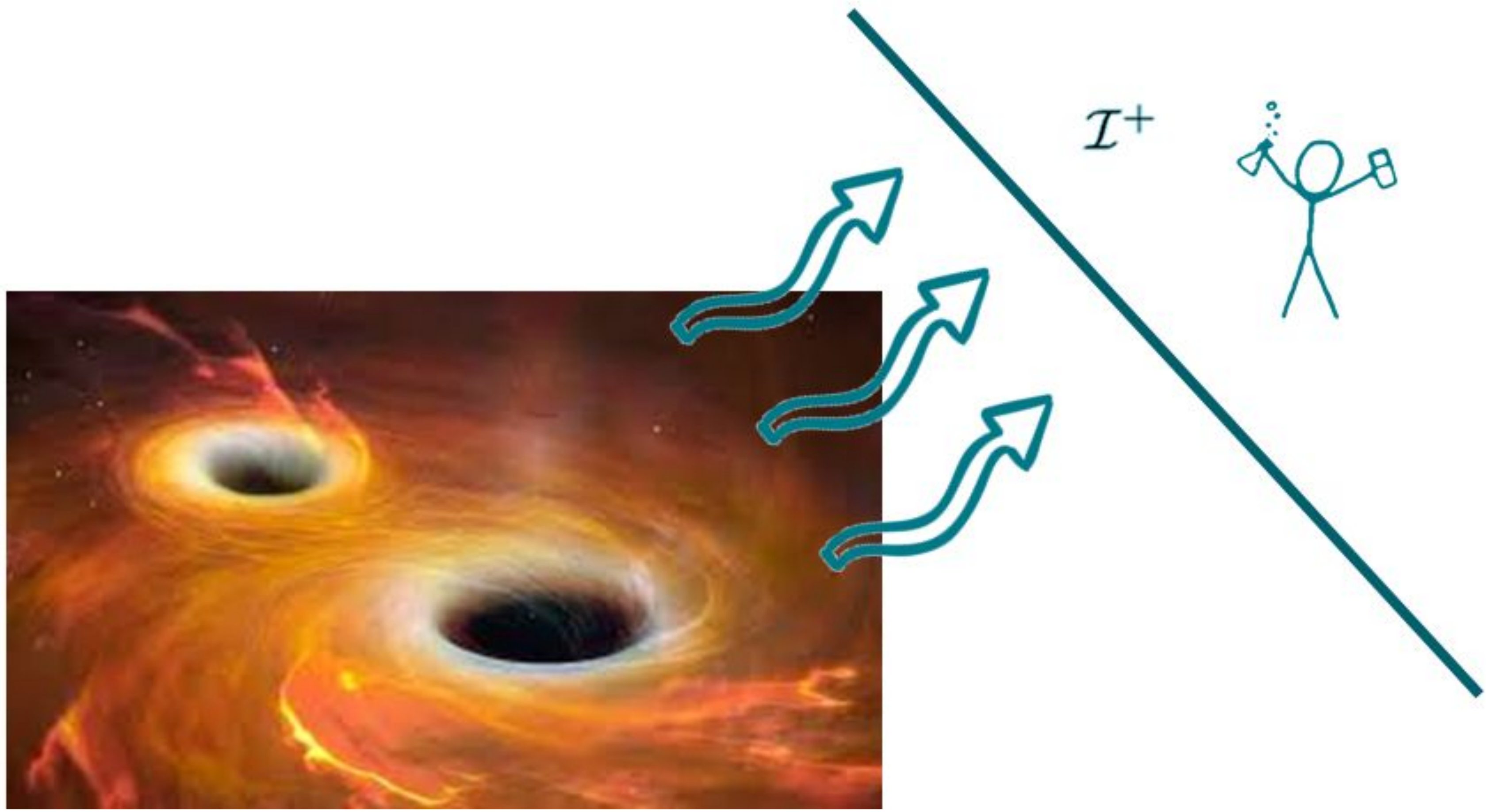
[BB+Prabhu, arXiv:2009.01243]

Overview

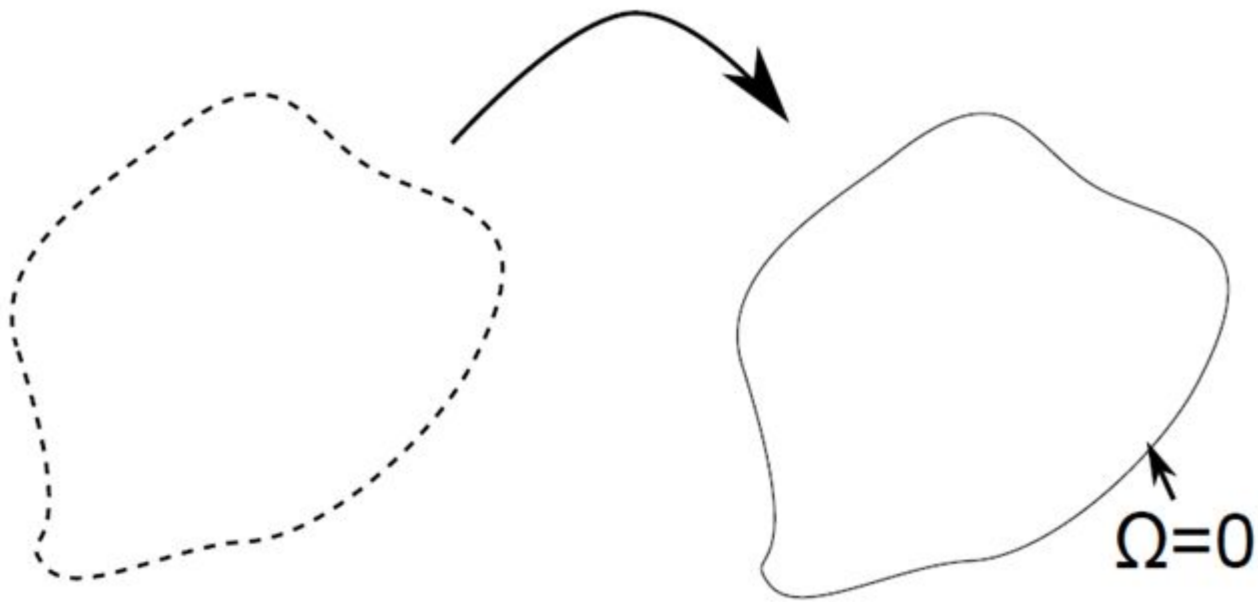
Asymptotic symmetries in cosmological spacetimes

but first some words about asymptotically flat spacetimes...

From messy physics to peaceful real



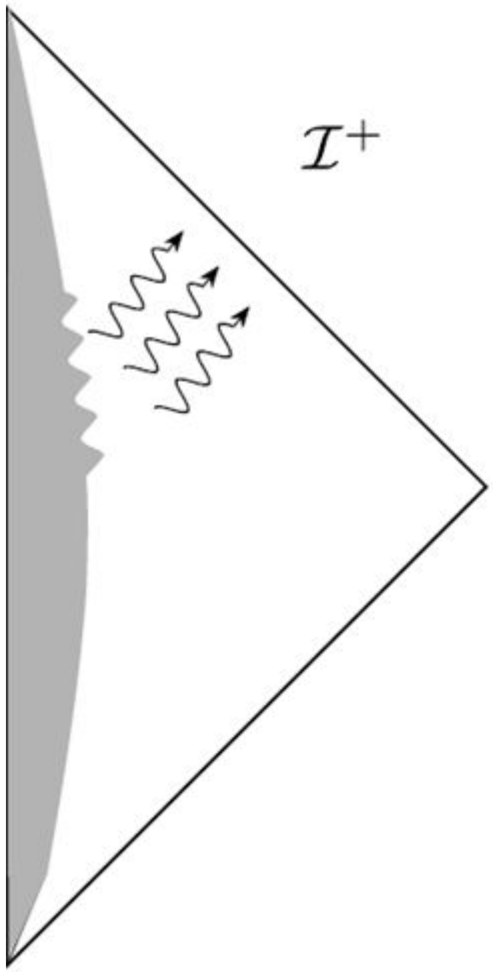
Key idea: bring infinity to a finite distance



$(\hat{M}, \hat{g}_{ab})$ $(M, g_{ab} = \Omega^2 \hat{g}_{ab})$

Conformal completion

Asymptotic flatness



A physical spacetime $(\widehat{M}, \hat{g}_{ab})$ is asymptotically flat if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{I} \cong \mathbb{R}\, x\, \mathbb{S}^2$ such that

1. Ω and $g_{ab} = \Omega^2 \; \hat{g}_{ab}$ are smooth on M, $\Omega \,\hat{=}\, 0$ and $n_a = \nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}$
2. Einstein's equations are satisfied with $\hat{T}_{ab}$ such that $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to $\mathcal{I}$

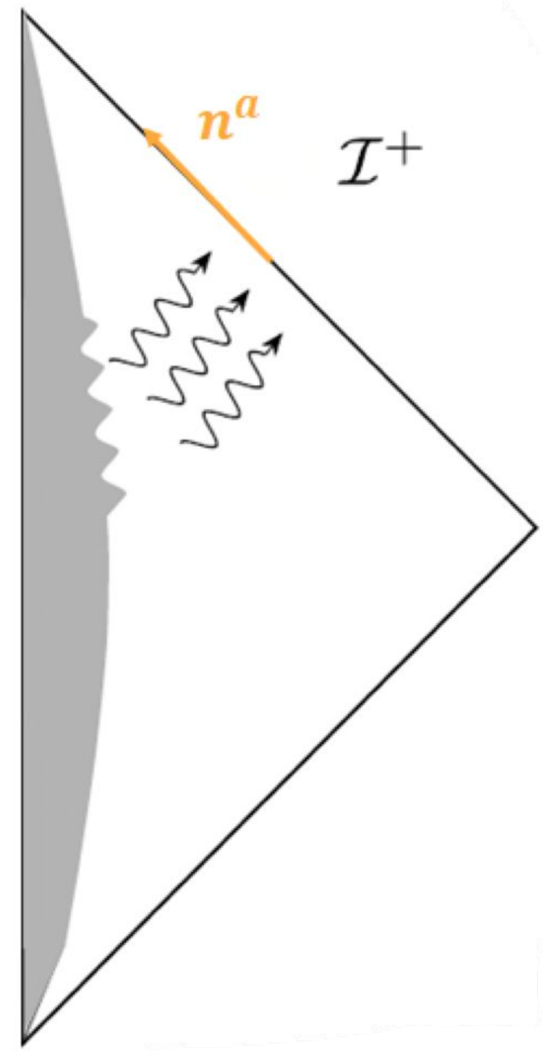
Consequences

- Einstein's equation $\Longrightarrow$ n^a is null on $\mathcal{J}$
- q_{ab} = induced metric on $\mathcal{J}$ is degenerate: 0 + +

Conformal freedom

$$\Omega \to \Omega' = \omega\, \Omega$$

$$g_{ab} \to g'_{ab} = \omega^2 g_{ab}$$
$$n^a \to n^{a\prime} = \omega^{-1} n^a$$



Universal structure

This is common to ***all*** asymptotically flat spacetimes

$$\{ q_{ab} , n^a \} = \{ \omega^2 q_{ab} , \omega^{-1} n^a \}$$

Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime

Example: flat spacetime

$$\Omega = \frac{1}{r}$$

$$dr = -\Omega^{-2} d\Omega$$

$$d\hat{s}^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \ d\phi^2\right)$$

$$d\hat{s}^2 = -du^2 - 2du\, dr + r^2 \left(d\theta^2 + \sin^2\theta \ d\phi^2\right)$$

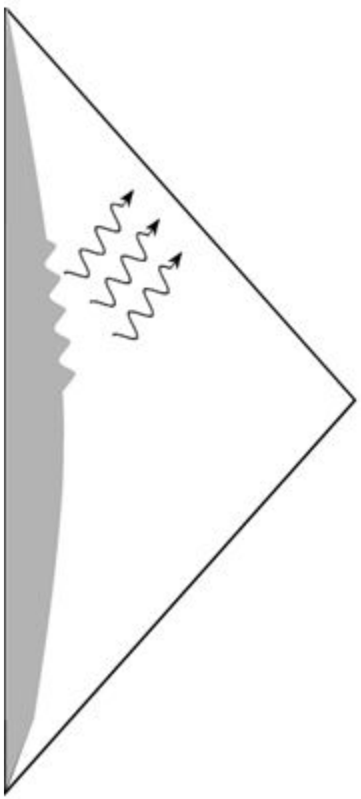
$$ds^2 = -\Omega^2\, du^2 + 2\, du\, d\Omega + d\theta^2 + \sin^2\theta \ d\phi^2$$

$$ds^2 \mathrel{\hat{=}} 2\, du\, d\Omega + d\theta^2 + \sin^2\theta \ d\phi^2$$

Others examples: Kerr-Newman spacetimes, Weyl spacetimes, etc.

Generic asymptotically flat spacetime

$$d\hat{s}^2 = -UV\,du^2 - 2U\,dudr + \gamma_{AB}(r\,d\theta^A + W^A\,du)(r\,d\theta^B + W^B\,du)$$



$$\begin{aligned}
U &= 1 + B/r^2 + O(r^{-3}),\\
V &= 1 - 2M/r + N/r^2 + O(r^{-3}),\\
W^A &= A^A/r + B^A/r^2 + O(r^{-3}),\\
\gamma_{AB} &= \Omega_{AB} + f_{AB}/r + \tfrac{1}{4}f^2\Omega_{AB}/r^2 + O(r^{-3})
\end{aligned}$$

Generic asymptotically flat spacetime

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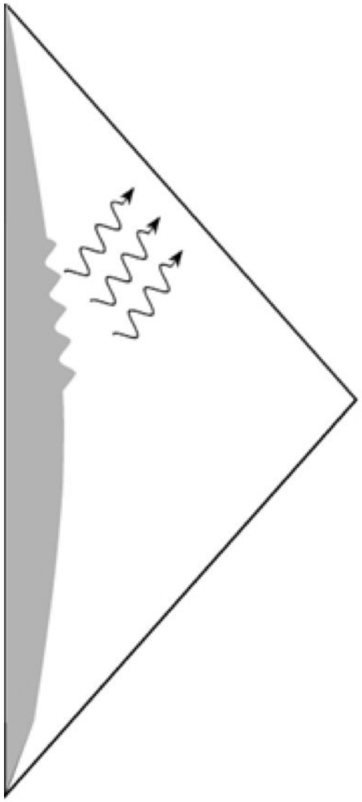
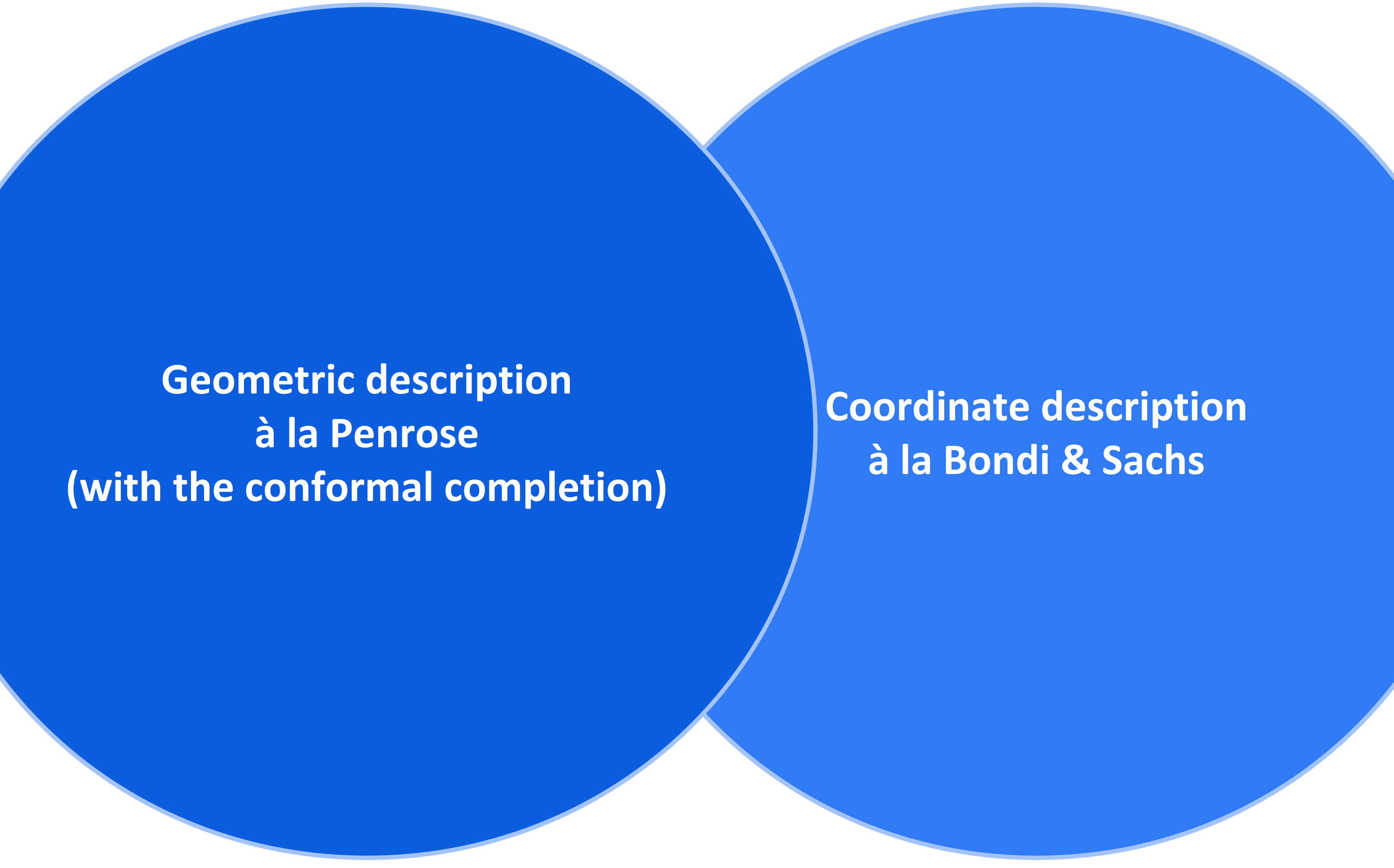


$$U = 1 + B/r^2 + O(r^{-3}),$$
$$V = 1 - 2M/r + N/r^2 + O(r^{-3}),$$
$$W^A = A^A/r + B^A/r^2 + O(r^{-3}),$$
$$\gamma_{AB} = \Omega_{AB} + f_{AB}/r + \tfrac{1}{4}f^2\Omega_{AB}/r^2 + O(r^{-3})$$

Flat Space | Mass aspect | Radiative modes | Angular momentum aspect

Two definitions are equivalent!



Asymptotic symmetry algebra

"Spacetime diffeomorphism that leave the universal structure at scri invariant"

Universal structure

$$\mathcal{L}_\xi q_{ab} \mathrel{\widehat{=}} 2\,\alpha\, q_{ab} \text{ with } \mathcal{L}_n \alpha \mathrel{\widehat{=}} 0$$

$$\mathcal{L}_\xi n^a \mathrel{\widehat{=}} -\,\alpha\, n^a$$

Coordinates

$$\Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \mathrel{\widehat{=}}\ 0$$

Bondi-Metzner-Sachs algebra (BMS)

- ***Bigger*** than Poincare (=translations & rotations)
- BMS = supertranslations & rotations

$$\xi^a \partial_a = \left(\boxed{f(\theta, \varphi)} + \frac{1}{2} \, u \, D_A Y^A \right) \partial_u + \boxed{Y^A \partial_A}$$

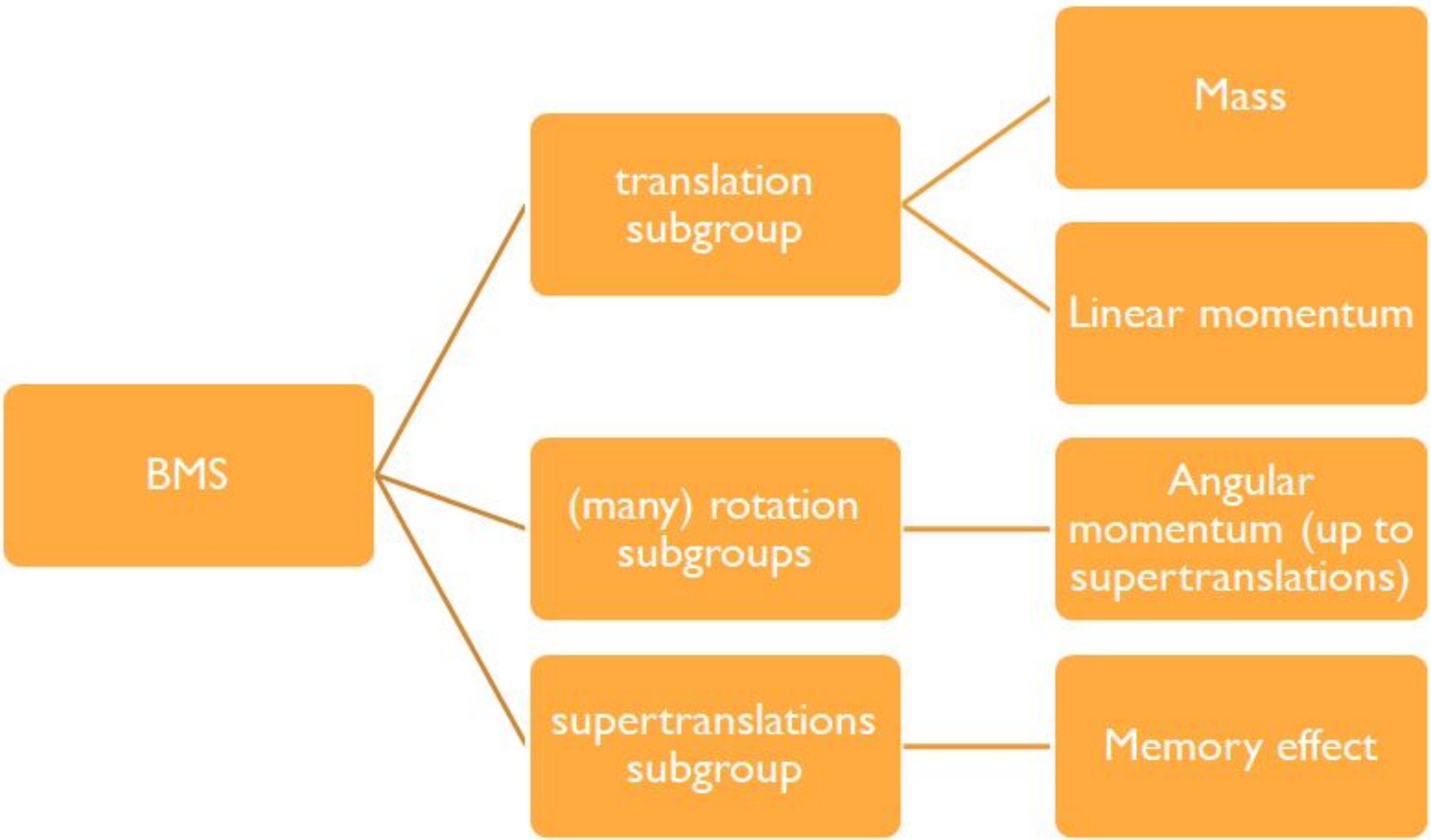
supertranslations

rotations

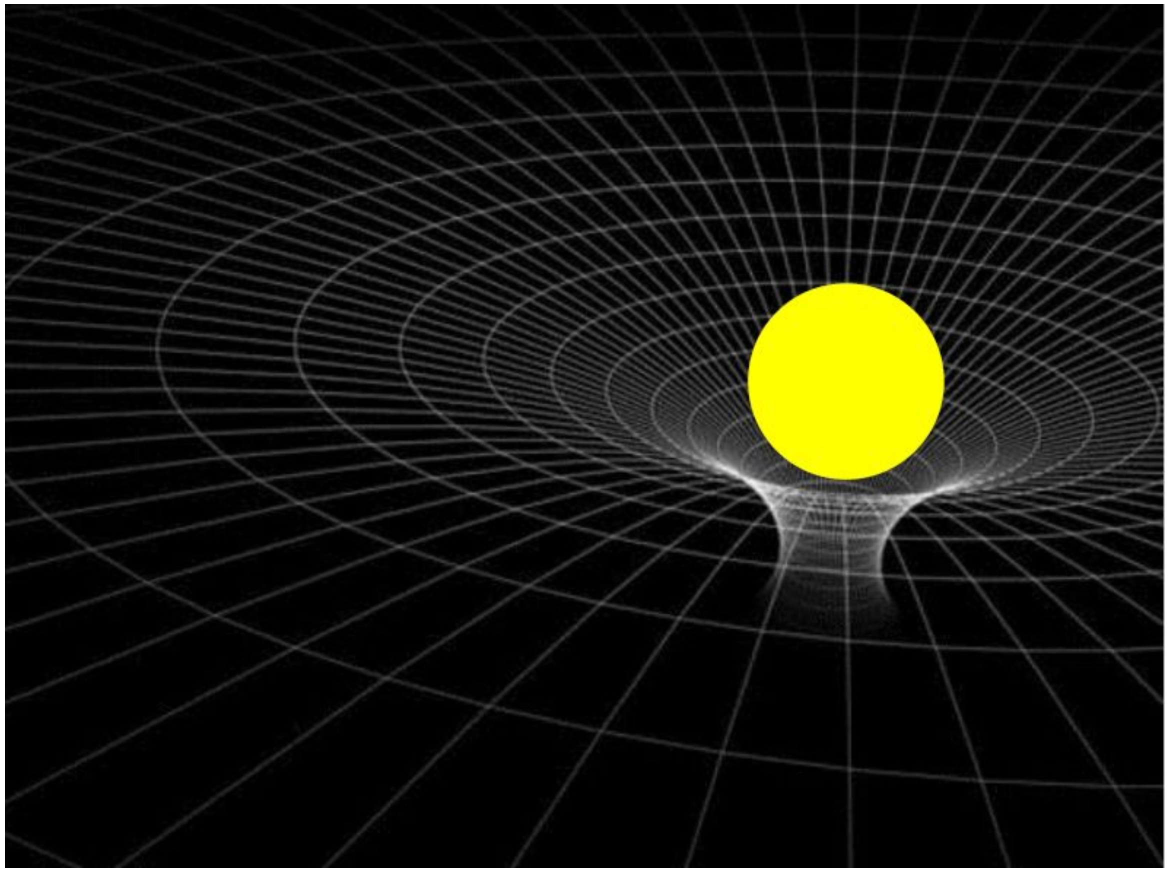
$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

What is BMS good for?

It provides quantities with a physical interpretation!

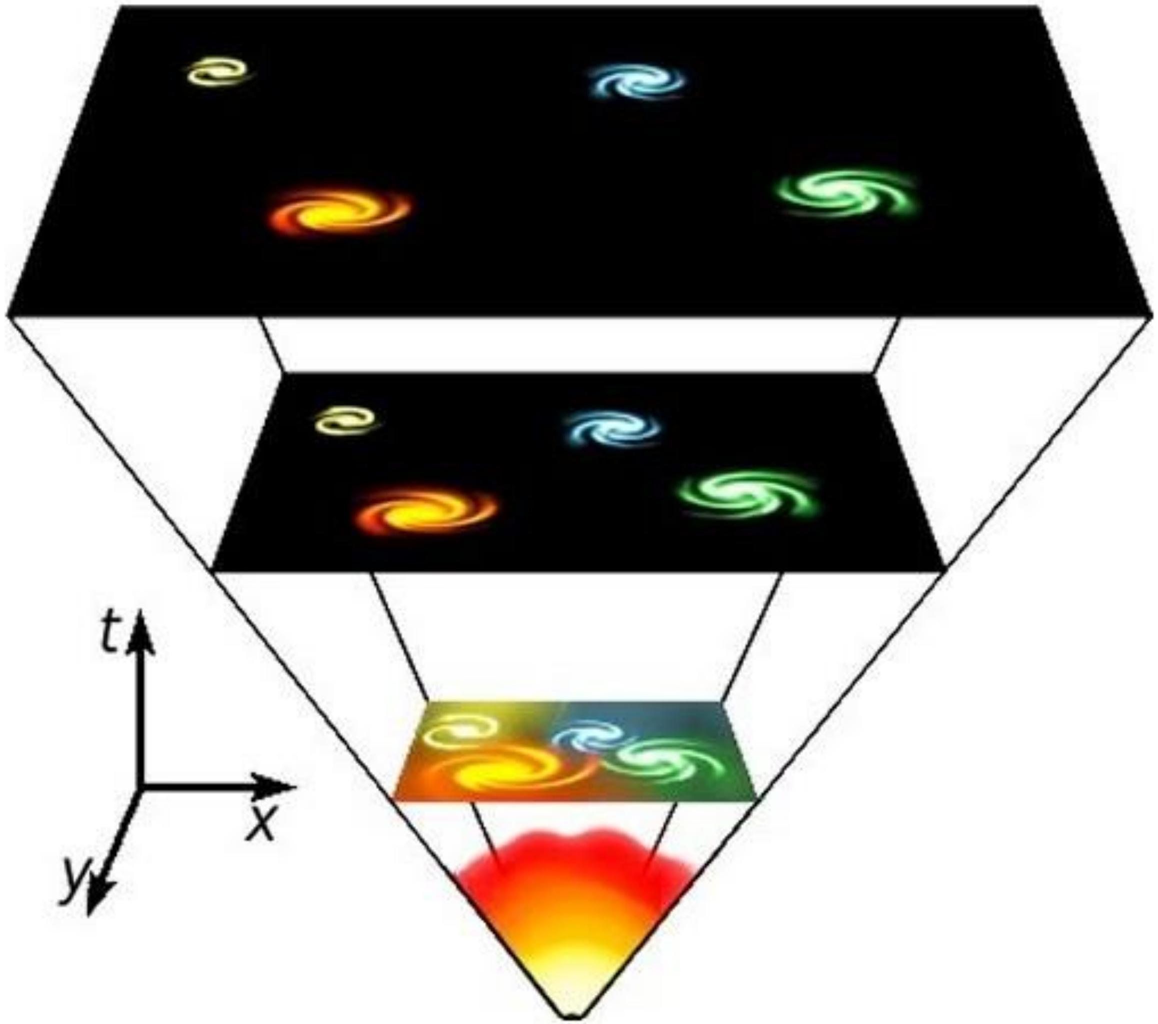


Critical assumption



Move far away from sources:
'spacetime becomes flat'

Expanding spacetimes are not asymptotically flat!



Why assume asymptotic flatness?

P. G. BERGMANN:

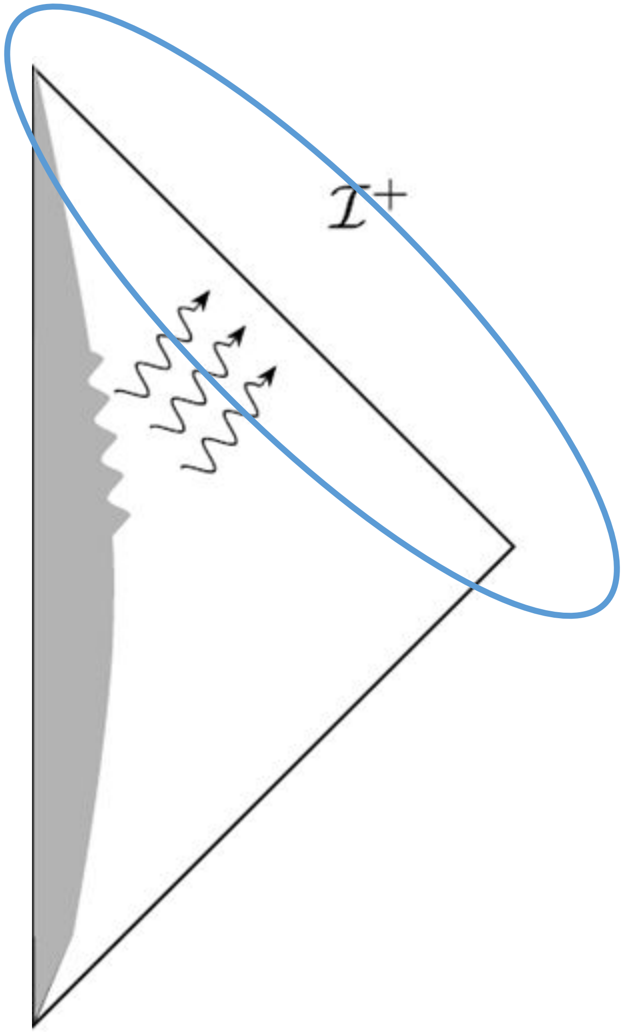
The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

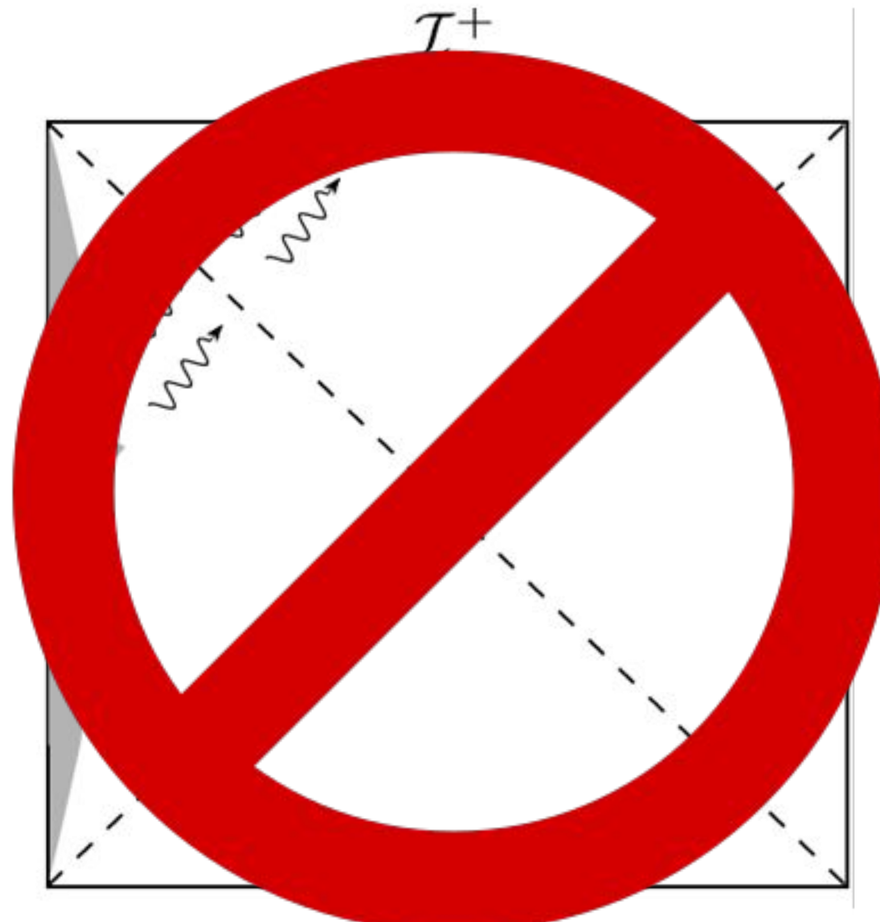
H. BONDI:

I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

Conference Warsaw 1963

Today: NO cosmological constant





Decelerating FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 S_{AB}\,dx^A dx^B\right]$$

physical metric

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{\frac{s}{1-s}}$$

$$s = \frac{2}{3(1+w)}$$

$$0 \leq s < 1$$

$$-1/3 < w < \infty$$

$$\underline{\underline{P = w\rho}}$$

$w = 1$	stiff fluid
$w = 1/3$	radiation
$w = 0$	dust
$w = -1$	cosmological constant

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 S_{AB}\, dx^A dx^B\right]$$

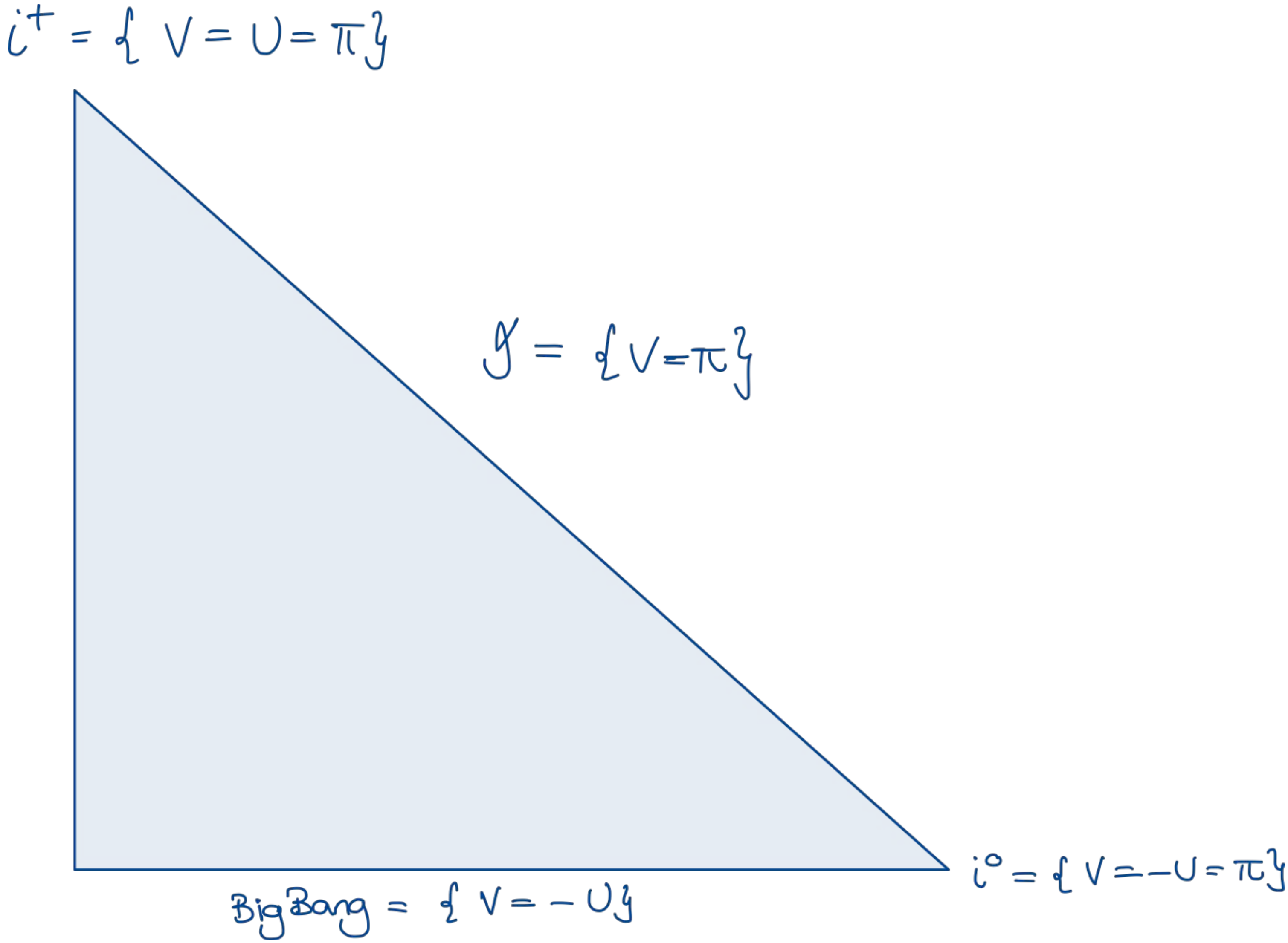
$$\eta = \frac{\sin T}{\cos R + \cos T} = \frac{\sin\left(\frac{V+U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}} \qquad r = \frac{\sin R}{\cos R + \cos T} = \frac{\sin\left(\frac{V-U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

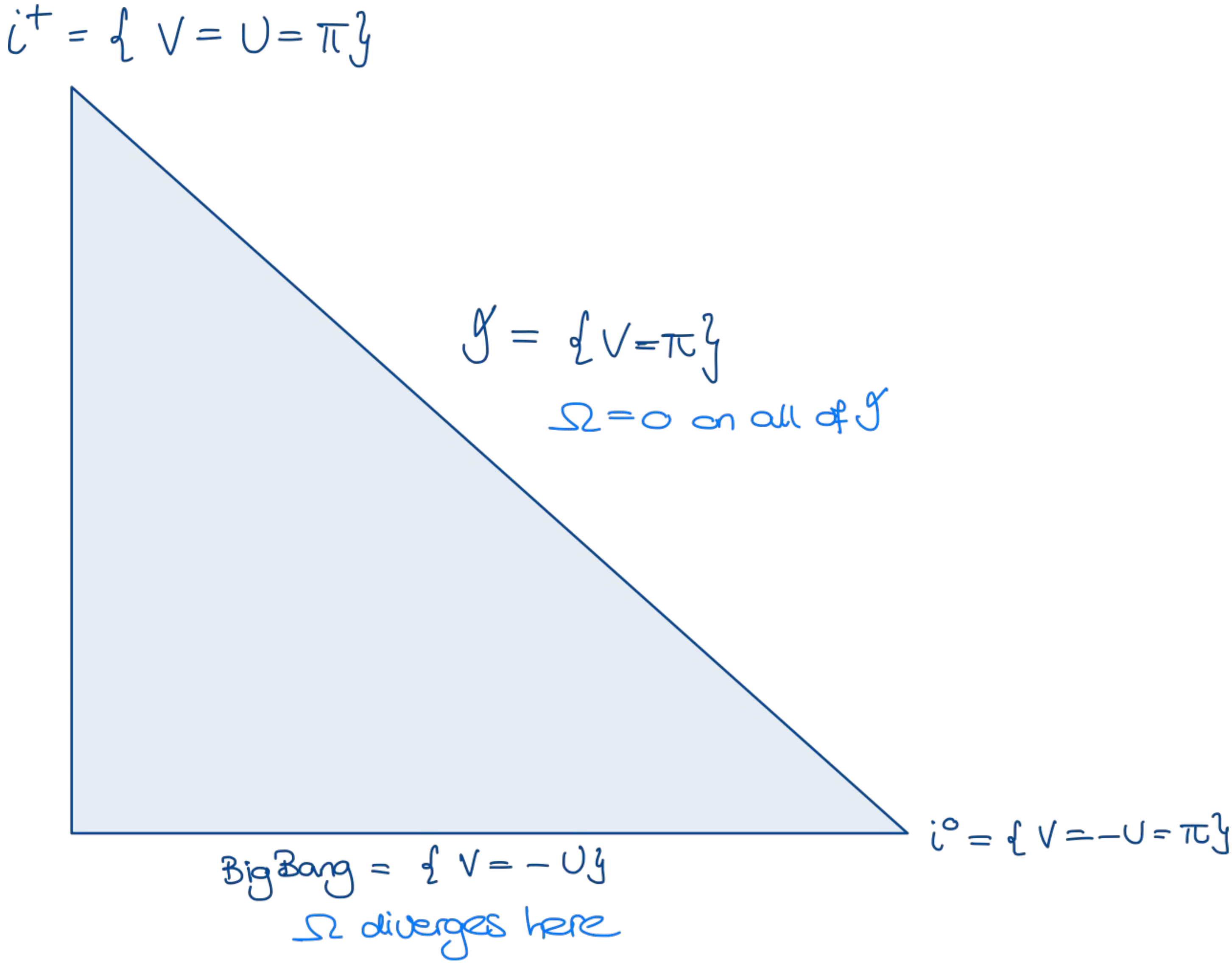
$$\begin{Bmatrix} U = T - R \\ V = T + R \end{Bmatrix} \Longleftrightarrow \begin{matrix} -\pi < U < \pi \\ |U| < V < \pi \end{matrix}$$

Choose $\Omega = 2\left(\cos\frac{U}{2}\cos\frac{V}{2}\right)^{\frac{1}{1-s}}\left(\sin\frac{U+V}{2}\right)^{-\frac{s}{1-s}}$

$$ds^2 = \Omega^2 d\hat{s}^2 = -dU\,dV + \sin\left(\frac{V-U}{2}\right)^2 S_{AB}\,dx^A dx^B$$

$\longrightarrow$ can add $V=-U$ & $V=\pi$, because this metric is smooth everywhere including at the boundaries


$i^+ = \{V = U = \pi\}$
$\mathscr{I} = \{V = \pi\}$
$i^0 = \{V = -U = \pi\}$
$\text{BigBang} = \{V = -U\}$



$i^+ = \{ V = U = \pi \}$
$\mathscr{I} = \{ V = \pi \}$
$\Omega = 0$ on all of $\mathscr{I}$
$i^0 = \{ V = -U = \pi \}$
BigBang $= \{ V = -U \}$
Ω diverges here

The conformal factor

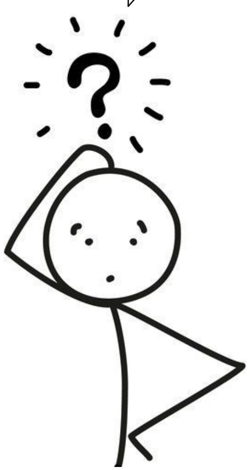
But near $\mathscr{I}$...

$$\Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{1}{1-s}} \longrightarrow \text{NOT smooth!}$$

$$\nabla_a \Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{s}{1-s}}\,\nabla_a V \longrightarrow \hat{=}\, 0 \text{ unless } s = 0$$

Bad choice for Ω?

What to do?



$$\Omega' = \omega\,\Omega \quad \text{with } \omega \sim (\pi - V)^{-\frac{s}{1-s}} \Longrightarrow$$

* Ω is smooth @ $\mathscr{I}$ ☑
* $\nabla_a \Omega \neq 0$ ☑

but then $g'_{ab} = \Omega'^2\, \hat{g}_{ab} \sim (\pi - V)^{-\frac{2s}{1-s}}\, \hat{g}_{ab}$

THIS DIVERGES @ $\mathscr{I}$!

Simple resolution

Ω^{1-s} is smooth @ γ

* $\Omega^{1-s} \hat{=} 0$
* $\nabla_a \Omega^{1-s} \hat{\neq} 0$

Define the normal to γ using Ω^{1-s}

$$\Rightarrow n_a := \frac{1}{1-s} \nabla_a \Omega^{1-s}$$

$$= \Omega^{-s} \nabla_a \Omega$$

$$\hat{=} -\frac{2^{-s}}{1-s} \left(\cos \frac{U}{2}\right)^{1-s} \nabla_a V$$

Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2}\hat{T}_{ab}$ should have a limit to $\mathcal{I}$ but FLRW spacetimes are homogeneous, so there is matter *everywhere*!

$$\lim_{\to \mathcal{I}} 8\pi G \, g^{ab}\hat{T}_{ab} = \frac{6s(1-s)}{(1-s)^2}\left(\sec\frac{U}{2}\right)^2 \longrightarrow \text{NON-VANISHING}$$

$$8\pi G \, \hat{T}_{ab} = \underbrace{2s\,\Omega^{2(s-1)}\, n_a n_b}_{\text{universal}} + 2s\,\Omega^{s-1}\,\tau_{(a} n_{b)} + \text{finite}$$

depends on choice Ω

$\tau_a \hat{=} \tan\frac{U}{2}\left(\nabla_a U + \nabla_a V\right)$

Spacetimes with a cosmological null asymptote

A physical spacetime $(\widehat{M}, \hat{g}_{ab})$ admits a cosmological null asymptote if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{I} \cong \mathbb{R}\, x\, \mathbb{S}^2$ such that

(1) $\Omega \mathrel{\widehat{=}} 0$, Ω^{1-s} and $g_{ab} = \Omega^2\ \hat{g}_{ab}$ is smooth on M,

$n_a = \Omega^{-s}\nabla_a\Omega$ is nowhere vanishing on $\mathcal{I}$ (for $0 \le s < 1$)

(2) Einstein's equations are satisfied with $\hat{T}_{ab}$ such that

$\lim_{\to \mathscr{I}} g^{ab}\hat{T}_{ab}$ exists

$$\lim_{\to \mathscr{I}} \Omega^{1-s}\left[8\pi\hat{T}_{ab} - 2s\Omega^{2(s-1)}n_a n_b\right] \mathrel{\widehat{=}} 2s\tau_{(a}n_{b)}$$

Spacetimes with a cosmological null asymptote

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$$\lim_{\to \mathscr{I}} g^{ab} \hat{T}_{ab} \quad \text{exists}$$

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Asymptotic symmetry algebra

All smooth vector fields that map

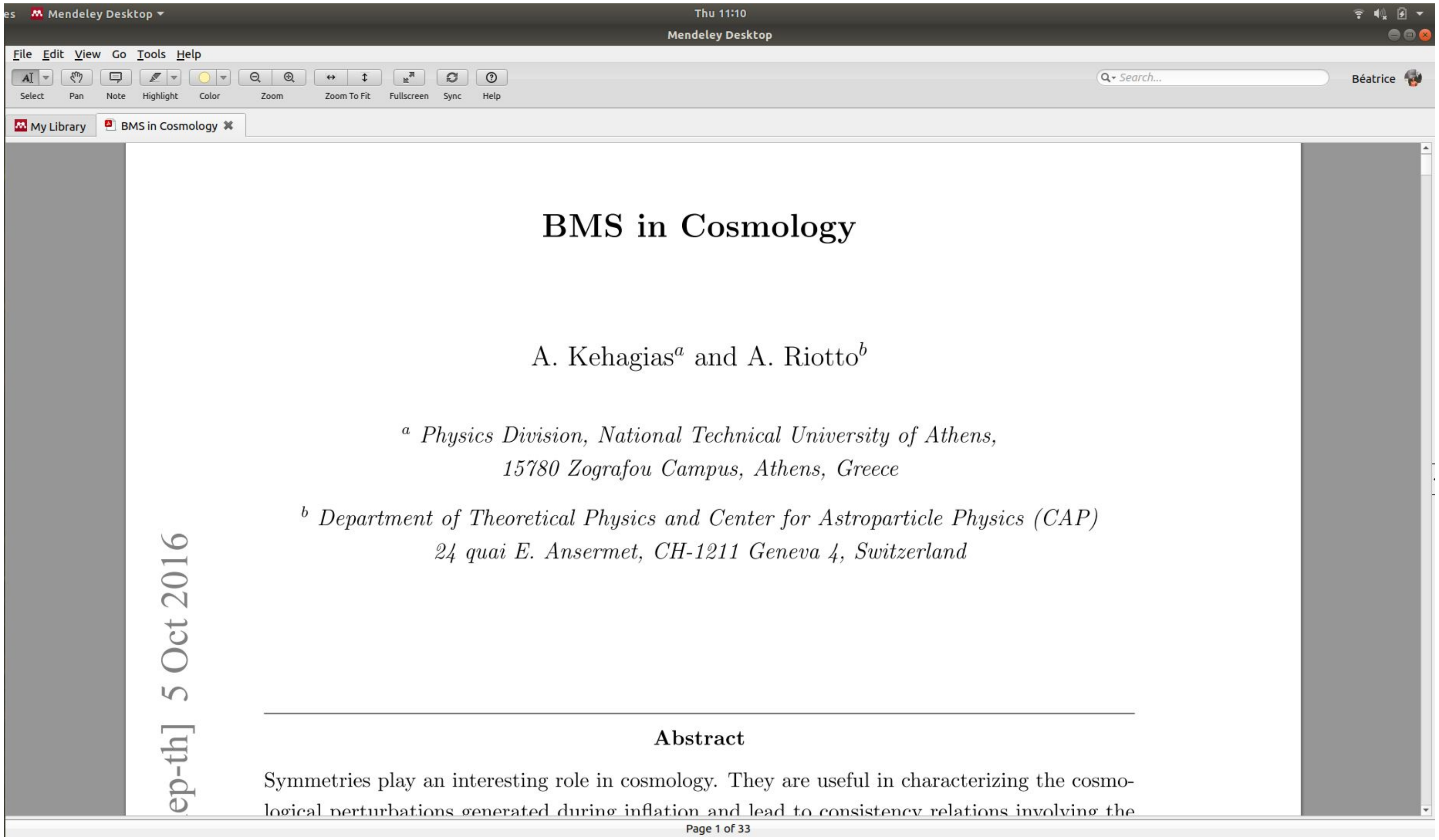
$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab},\ n'^a = \omega^{-1-s} n^a\}$$

$$\boxed{\Longrightarrow \mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathfrak{S}_s}$$

Lorentz subalgebra

s-dependent supertranslations

Didn’t we know this already?



There is a twist!

Supertranslations

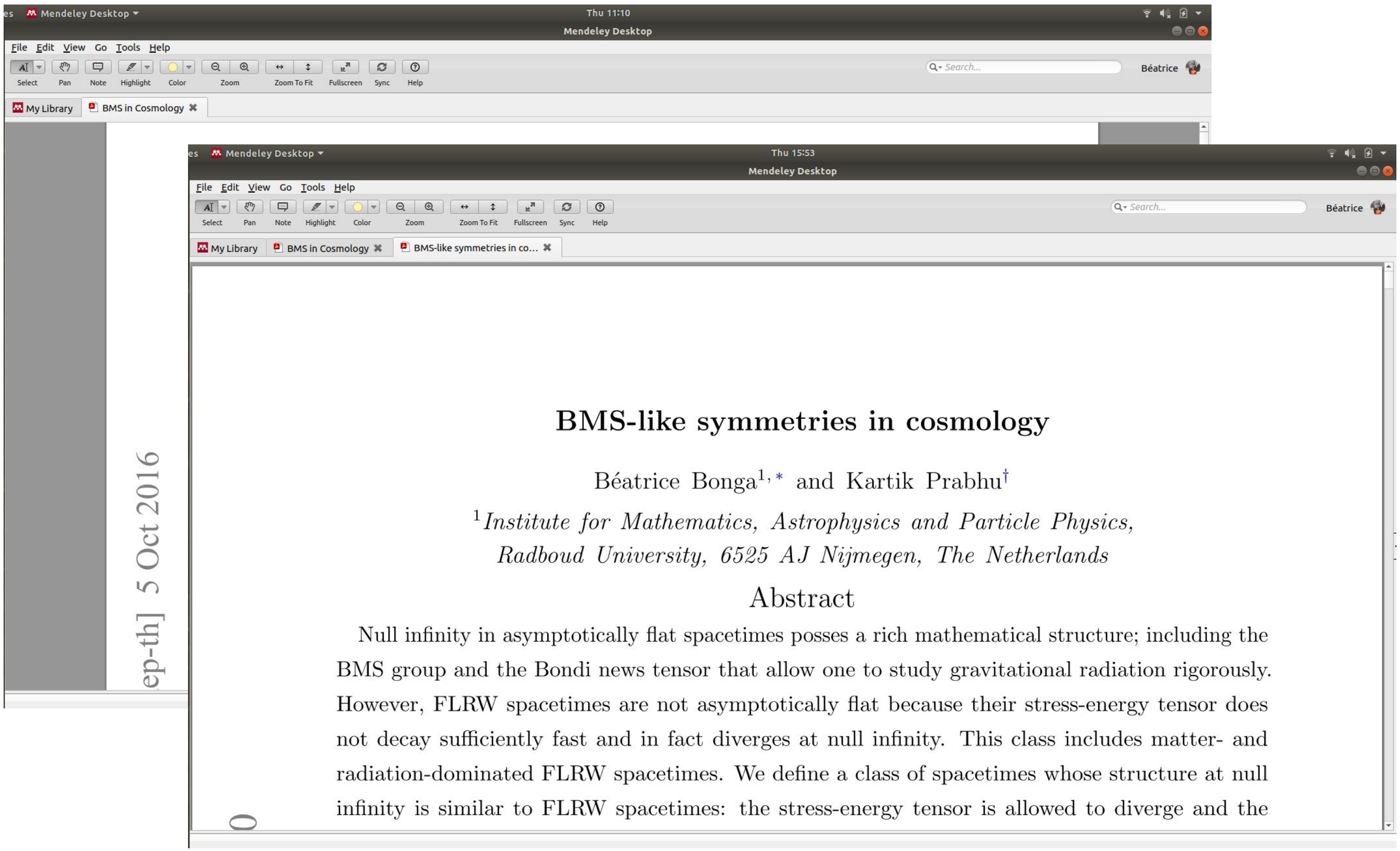
$$\xi^a = f(\theta,\varphi)\, n^a$$

f has conformal weight $1+s$

Who cares?

No translation subalgebra

Not exactly BMS in cosmology



BMS-like symmetries in cosmology

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Abstract

Null infinity in asymptotically flat spacetimes posses a rich mathematical structure; including the BMS group and the Bondi news tensor that allow one to study gravitational radiation rigorously. However, FLRW spacetimes are not asymptotically flat because their stress-energy tensor does not decay sufficiently fast and in fact diverges at null infinity. This class includes matter- and radiation-dominated FLRW spacetimes. We define a class of spacetimes whose structure at null infinity is similar to FLRW spacetimes: the stress-energy tensor is allowed to diverge and the

notion of mass and linear momentum?

Any other examples?

Class of spacetimes at least as big as asymptotically flat spacetimes!

Linearization stability still open question

Conclusion

- ❖ Geometric construction to study spacetimes beyond asymptotic flatness in the cosmological context

- ❖ Asymptotic symmetry algebra is BMS-***like***

  - ➢ It does not have a translation subalgebra!

Future applications

- Next order structure
  - Study rigorously gravitational radiation produced by compact sources in cosmological spacetimes
  - Study the gravitational memory effect
  - Charges and fluxes
- Link with timelike future infinity
- … your favorite topic!